Discrete logarithm algorithms in pairing-relevant finite fields

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The discrete logarithm problem (DLP)

Asymmetric cryptography relies on the hardness of either factorization (RSA) or the **discrete logarithm problem.**

→ Used in Diffie-Hellman, El-Gamal, (EC)DSA, etc

Definition

Given a finite cyclic group G, a generator $g \in G$ and a target $h \in G$, find x such that $h = g^x$.

Commonly used groups: prime finite fields $\mathbb{F}_p^* = (\mathbb{Z}/p\mathbb{Z})^*$, finite fields $\mathbb{F}_{p^n}^*$, elliptic curves over finite fields $\mathcal{E}(\mathbb{F}_p)$ ...

Groups G for which DLP is hard

Examples in the wild

Widely deployed protocols base their security on the hardness of DLP on a group G.

Ephemeral Diffie Hellman

Technical Details

Connection Encrypted (TLS_ECDHE_RSA_WITH_AES_128_GCM_SHA256, 128 bit keys, TLS 1.2)

An interesting example: pairing-based protocols!

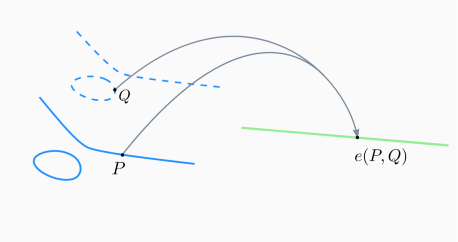


Fig from Diego Aranha

Pairing-based cryptography

What is a cryptographic pairing ?

- $\mathbb{G}_1, \mathbb{G}_2$: additive groups of prime order ℓ.
- $\mathbb{G}_T$: multiplicative group of prime order ℓ.

A pairing is a map $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$

- with bilinearity: $\forall a, b \in \mathbb{Z}, e(aP, bQ) = e(P, Q)^{ab}$,
- non-degeneracy: $\exists P, Q$ such that $e(P, Q) \neq 1$,
- and such that e is efficiently computable (for practicality reasons).

Called symmetric if $\mathbb{G}_1 = \mathbb{G}_2$.

Security of pairing-based protocols

Most of the time, in cryptography:

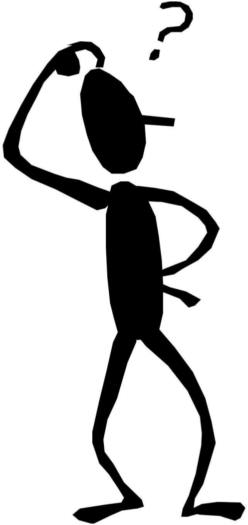
- $\mathbb{G}_1$ = subgroup of $\mathcal{E}(\mathbb{F}_p)$,
- $\mathbb{G}_2$ = subgroup of $\mathcal{E}(\mathbb{F}_{p^n})$,
- $\mathbb{G}_T$ = subgroup of finite field $\mathbb{F}_{p^n}^*$.

Why do we care ? hundreds of old and many recent protocols built with pairings.
Example: zk-SNARKS (blockchain, Zcash ...)

⟶ Example that uses DLP on both elliptic curves and finite fields.

Question: **How to construct a secure pairing-based protocol ?**
⟶ Look at DLP algorithms on both sides!

The discrete logarithm problem in finite fields



- Many different algorithms for DLP in $\mathbb{F}_{p^n}$
- Their complexity depends on the relation between characteristic p and extension degree n.

Useful notation

→ Complexity depends on the relation between characteristics p and extension degree n.

L-notation:

$$L_{p^n}(l_p, c) = \exp((c + o(1))(\log(p^n))^{l_p}(\log\log p^n)^{1-l_p}),$$

for $0 \leqslant l_p \leqslant 1$ and some constant $c > 0$.

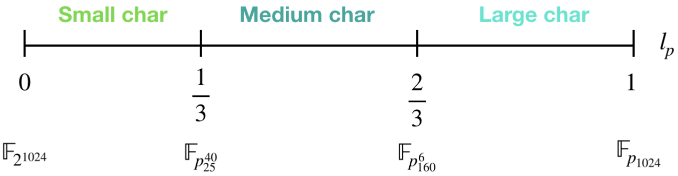
For complexities:

- When $l_p \to 0$: $\exp\left(\log\log p^n\right) \approx \log p^n$ Polynomial-time
- When $l_p \to 1$: p^n Exponential-time

In the middle, we talk about subexponential time.

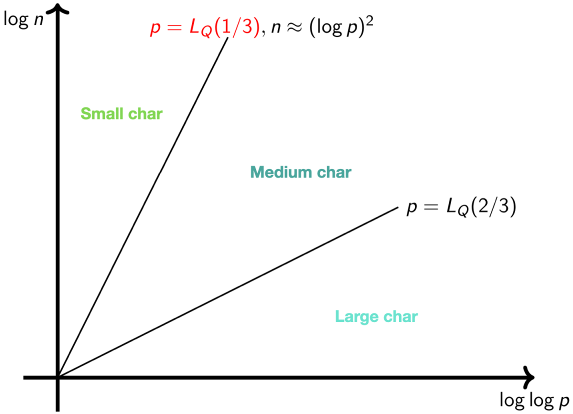
Three families of finite fields

Finite field: $\mathbb{F}_{p^n}$, with $p = L_{p^n}(l_p, c_p)$



- Different algorithms are used in the different zones.
- Algorithms don't have the same complexity in each zone.

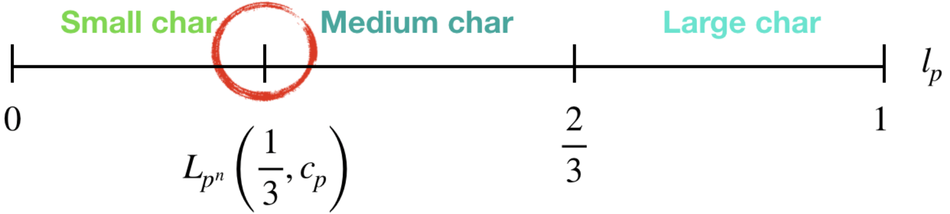
The L-notation for $\mathbb{F}_Q$ with $Q = p^n$



Question: Which area do we focus on ?

The first boundary case

In this work, we focus on the boundary case $p = L_{p^n}(1/3)$, the area between the small and the medium characteristics.



Why?

1. Area where pairings take their values.
2. Many algorithms overlap: $\longrightarrow$ which algorithm has the lowest complexity ?

Balancing complexities for the security of pairings

Idea: For pairings, we want DLP to be as hard on the elliptic curve side than on the finite field side.

- choose the area where DLP in finite fields is the most difficult;

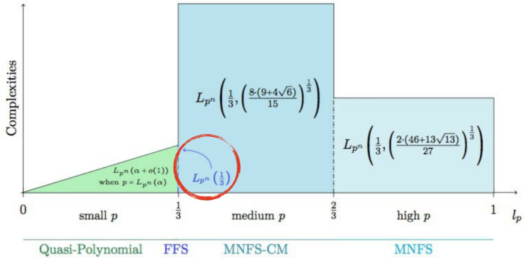
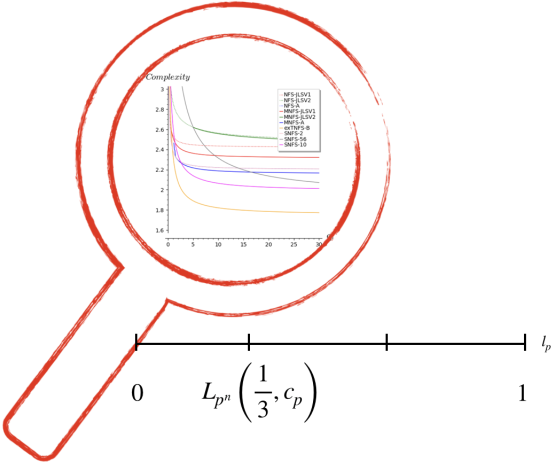


Fig. Cécile Pierrot

- "balance" complexity on elliptic curves and finite fields:

$$\sqrt{p} = L_{p^n}(1/3) \Rightarrow p = L_{p^n}(1/3)$$

Main results of the paper



- Analyse the behaviour of many algorithms in this area.
- Estimate the security of pairing-based protocols.

Consider a finite field $\mathbb{F}_{p^n}$.
Factor basis: $\mathcal{F}$ = small set of " small " elements.
Three main steps:

1. Relation collection: find relations between the elements of $\mathcal{F}$.
2. Linear algebra: solve a system of linear equations where the unknowns are the discrete logarithms of the elements of $\mathcal{F}$.
3. Individual logarithm: for a target element $h \in \mathbb{F}_{p^n}$, compute the discrete logarithm of h.

The Number Field Sieve and its variants are examples of *index calculus algorithms.*

The complexity of NFS and its variants

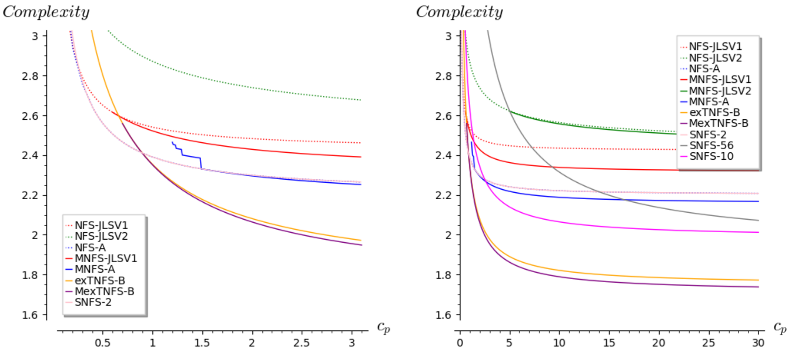
- 3 phases = 3 costs $\longrightarrow$ overall complexity is sum of 3 costs.

Goal: Optimize the maximum of these three costs.

Why complicated?

1. Many parameters $\longrightarrow$ discrete or continuous, boundary issues.
2. Optimization problem $\longrightarrow$ Lagrange multipliers.
3. Solving a polynomial system $\longrightarrow$ Gröbner basis algorithm.
4. Uses many analytic number theory results.

A summary of these complexities

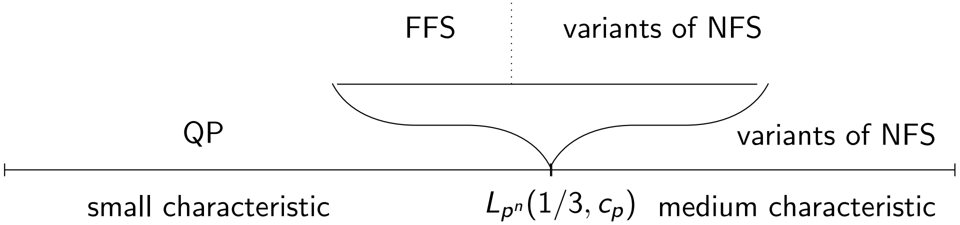


Recall $p = L_{p^n}(1/3, c_p)$, and complexities $= L_{p^n}(1/3, f(c_p))$

Surprising fact:

- Not all the variants are applicable at the boundary case: STNFS has a much higher complexity!

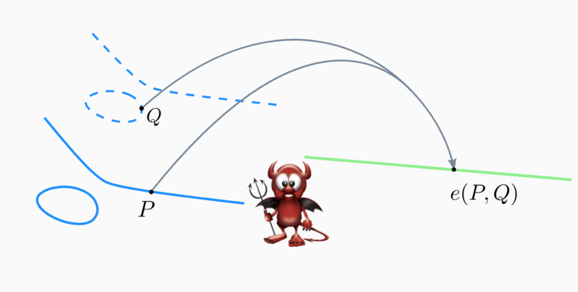
And the winners are ... !



For the variants of NFS, the best algorithm depends on considerations on n and p.

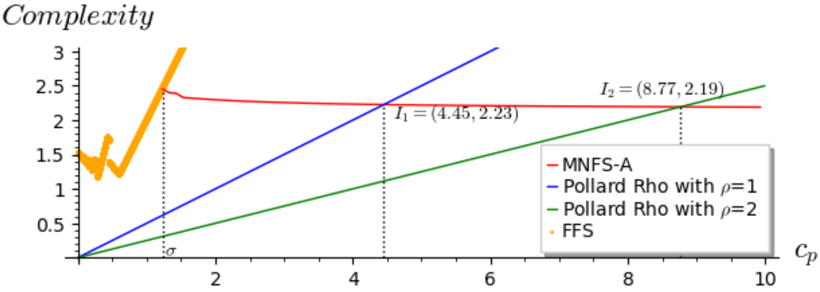
Constructing secure pairings

Asymptotically what finite field $\mathbb{F}_{p^n}$ should be considered in order to achieve the highest level of security when constructing a pairing?



Goal: find the optimal p and n that answers this question.

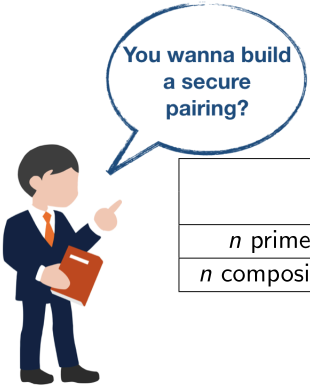
Goal: Look for value of c_p that maximizes $\min(\text{comp}_{\mathcal{E}}, \text{comp}_{\mathbb{F}_{p^n}})$.



- Complexities for finite field DLP are decreasing functions.
- Pollard rho is an increasing function ($\text{complexity}_{\mathcal{E}} = p^{1/2\rho}$)

→ optimal c_p given by the intersection point!

Conclusion for pairings



	normal p	special p $\lambda = 20$	special p $\lambda = 3$
n prime	$c_p = 4.45, \; c_{\textbf{MNFS-}\mathcal{A}} = 2.23$		$c_p = 4.36, \; c_{\textbf{SNFS-3}} = 2.18$
n composite	$c_p = 3.91, \; c_{\textbf{MexTNFS-B}} = 1.91$		

Suprising fact: Using a special form for p does not always make the pairing less secure ! It depends on the value of λ.

Thank you for your attention!

Questions?