Discrete logarithm algorithms in pairing-relevant finite fields

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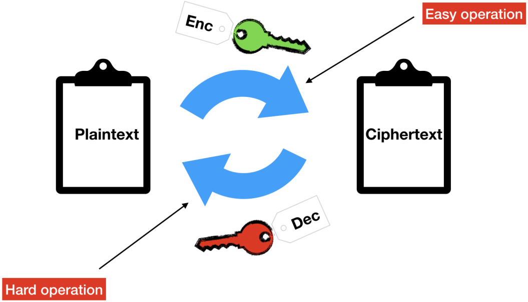
Asymmetric cryptography



Relies on the hardness of two main mathematical problems:

- Factorization (RSA cryptosystem)
- Discrete logarithm problem

The discrete logarithm problem (DLP)

$\longrightarrow$ Used in Diffie-Hellman, El-Gamal, (EC)DSA, etc

Definition

Given a finite cyclic group G, a generator $g \in G$ and a target $h \in G$, find x such that $h = g^x$.

$\boxed{\textbf{Which group } G \textbf{ should we consider ?}}$

Groups for DLP

In cryptography, choose G such as DLP is difficult:

- prime finite fields $\mathbb{F}_p^* = (\mathbb{Z}/p\mathbb{Z})^*$,
- class groups of number fields,
- finite fields $\mathbb{F}_{p^n}^*$,
- elliptic curves over finite fields $\mathcal{E}(\mathbb{F}_p)$,
- genus 2 hyperelliptic curves.

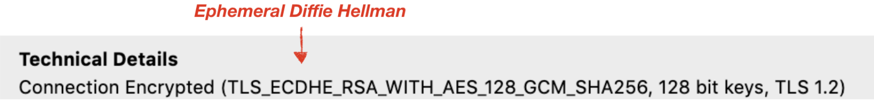
One bad idea: $(\mathbb{Z}/N\mathbb{Z}, +)$ where DLP is simply a division.

Classical assumptions:

- The order of the group is known.
- There exists an efficient algorithm for the group law.

Examples in the wild

Widely deployed protocols base their security on the hardness of DLP on a group G.



An interesting example: pairing-based protocols!

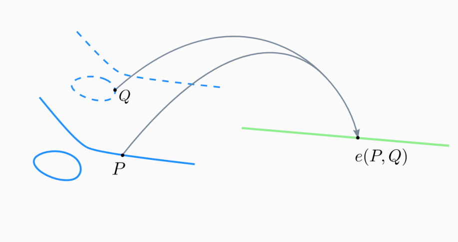


Fig from Diego Aranha

Pairing-based cryptography

What is a cryptographic pairing ?

- $\mathbb{G}_1, \mathbb{G}_2$: additive groups of prime order ℓ.
- $\mathbb{G}_T$: multiplicative group of prime order ℓ.

$$\boxed{\textbf{A pairing is a map } e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T}$$

- with bilinearity: $\forall a, b \in \mathbb{Z}, e(aP, bQ) = e(P, Q)^{ab}$,
- non-degeneracy: $\exists P, Q$ such that $e(P, Q) \neq 1$,
- and such that e is efficiently computable (for practicality reasons).

Called symmetric if $\mathbb{G}_1 = \mathbb{G}_2$.

Security of pairing-based protocols

Most of the time, in cryptography:

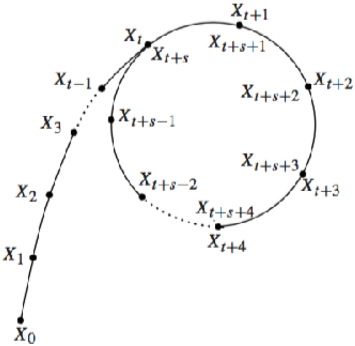
- $\mathbb{G}_1 =$ subgroup of $\mathcal{E}(\mathbb{F}_p)$,
- $\mathbb{G}_2 =$ subgroup of $\mathcal{E}(\mathbb{F}_{p^n})$,
- $\mathbb{G}_T =$ subgroup of finite field $\mathbb{F}_{p^n}^*$.

Why do we care ? hundreds of old and many recent protocols built with pairings.
Example: zk-SNARKS (blockchain, Zcash ...)

$\longrightarrow$ Example that uses DLP on both elliptic curves and finite fields.

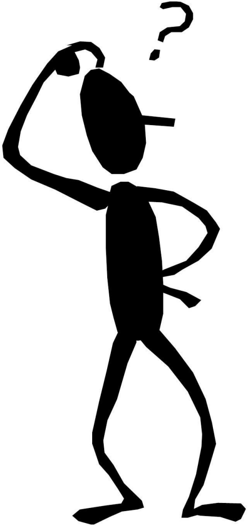
Question: **How to construct a secure pairing-based protocol ?**
$\longrightarrow$ Look at DLP algorithms on both sides!

The discrete logarithm problem on elliptic curves



- Best algorithm: Pollard Rho
- Complexity: square root of the size of the subgroup considered.
- No gain except for constant factor since the 70s.

The discrete logarithm problem in finite fields



- Many different algorithms for DLP in $\mathbb{F}_{p^n}$
- Their complexity depends on the relation between characteristic p and extension degree n.

Useful notation

$\longrightarrow$ Complexity depends on the relation between characteristics p and extension degree n.

L-notation:

$$\boxed{L_{p^n}(l_p, c) = \exp((c + o(1))(\log(p^n))^{l_p}(\log\log p^n)^{1-l_p}),}$$

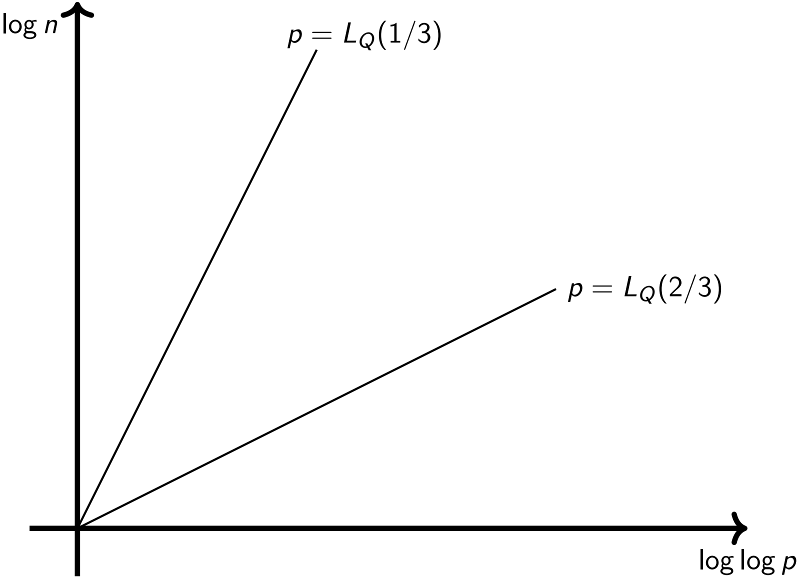
for $0 \leqslant l_p \leqslant 1$ and some constant $c > 0$.

For complexities:

- When $l_p \to 0$: $\exp\left(\log\log p^n\right) \approx \log p^n$ Polynomial-time
- When $l_p \to 1$: p^n Exponential-time

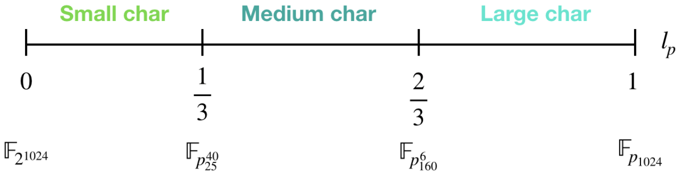
In the middle, we talk about subexponential time.

The L-notation for $\mathbb{F}_Q$ with $Q = p^n$ Slide from Pierrick Gaudry



Three families of finite fields

Finite field: $\mathbb{F}_{p^n}$, with $p = L_{p^n}(l_p, c_p)$

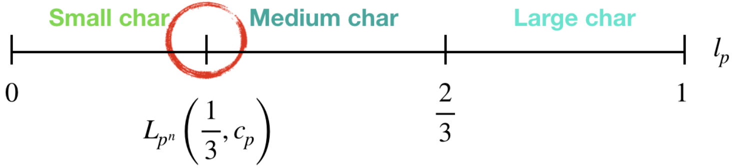


- Different algorithms are used in the different zones.
- Algorithms don't have the same complexity in each zone.

Question: Which area do we focus on ?

The first boundary case

In this work, we focus on the boundary case $p = L_{p^n}(1/3)$, the area between the small and the medium characteristics.



Why?

1. Area where pairings take their values.
2. Many algorithms overlap: $\longrightarrow$ which algorithm has the lowest complexity ?

Balancing complexities for the security of pairings

Idea: For pairings, we want DLP to be as hard on the elliptic curve side than on the finite field side.

- choose the area where DLP in finite fields is the most difficult;

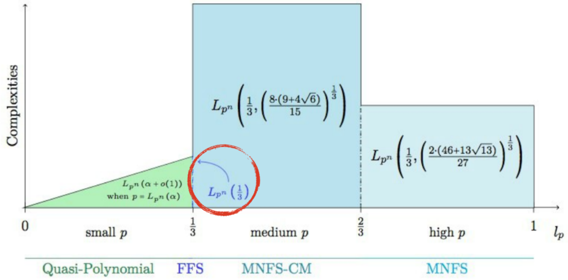
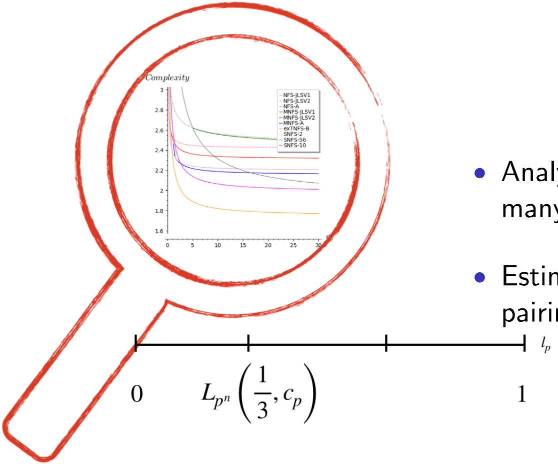


Fig. Cécile Pierrot

- "balance" complexity on elliptic curves and finite fields:

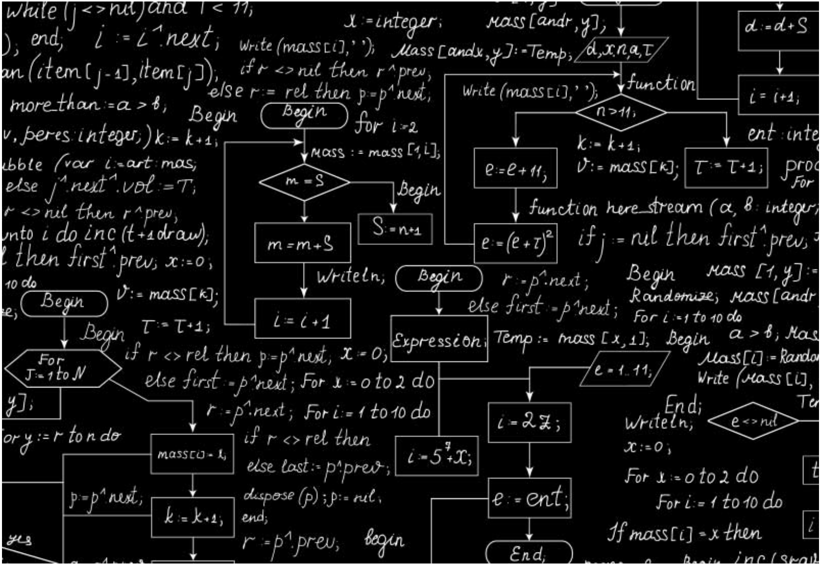
$$\boxed{\sqrt{p} = L_{p^n}(1/3) \Rightarrow p = L_{p^n}(1/3)}$$

The road ahead



- Analyse the behaviour of many algorithms in this area.
- Estimate the security of pairing-based protocols.

Index Calculus Algorithms



The index calculus algorithms

Consider a finite field $\mathbb{F}_{p^n}$.
Factor basis: $\mathcal{F}$ = small set of " small " elements.
Three main steps:

1. Relation collection: find relations between the elements of $\mathcal{F}$.
2. Linear algebra: solve a system of linear equations where the unknowns are the discrete logarithms of the elements of $\mathcal{F}$.
3. Individual logarithm: for a target element $h \in \mathbb{F}_{p^n}$, compute the discrete logarithm of h.

The Number Field Sieve

$$\begin{tikzcd}[column sep=small]
 & & a - bx \quad \mathbb{Z}[X] \arrow[dl, "X\mapsto\theta_1"'] \arrow[dr, "X\mapsto\theta_2"] & & \\
\begin{array}{c} a - b\theta_1 \\ N_1(a - b\theta_1) \\ \textbf{B-smooth?} \end{array} & \mathbb{Q}[X]/(f_1(X)) \arrow[dr, "\theta_1\mapsto m"'] & & \mathbb{Q}[X]/(f_2(X)) \arrow[dl, "\theta_2\mapsto m"] & \begin{array}{c} a - b\theta_2 \\ N_2(a - b\theta_2) \\ \textbf{B-smooth?} \end{array} \\
 & & \mathbb{F}_{p^n} & &
\end{tikzcd}$$

1. f_1, f_2 irreducible in $\mathbb{Z}[X]$ s.t. the diagram commutes.
2. Compute the algebraic norms in $\mathbb{Z}$: $N(a - b\theta_i)$
3. Factor $N_i(a - b\theta_i)$ in $\mathbb{Z}$ into prime numbers
4. If prime factors $\leqslant B$ on both sides $\longrightarrow$ relation

Collecting relations, solving a system...

A relation in $\mathbb{F}_{p^n}$ implies the equality:

$$a - b\theta_1 \text{``}=\text{''} \prod_{f \in \mathcal{F}} f^{\alpha_i} \equiv \prod_{f \in \mathcal{F}} f^{\beta_i} \text{``}=\text{''} a - b\theta_2.$$

Take the discrete logarithm on both sides:

$$\sum_{f \in \mathcal{F}} \alpha_i \log f = \sum_{f \in \mathcal{F}} \beta_i \log f \pmod{p^n - 1}$$

$=$ **linear relation between log elements of the factor basis $\mathcal{F}$.**

Goal: Get as many equations/relations of log of elements of the factor basis.

Why? we want to solve a linear system!

Solving the linear system and a descent phase

Linear algebra:

- unknowns are the $\log f$ for $f \in \mathcal{F}$.
- solve the system to recover the values $\log f$.

How do we solve the system? Sparse linear algebra algorithms : block Wiedemann algorithm in $O(k^2)$, where k is the size of the system.

Descent phase: our target is $h \in \mathbb{F}_{p^n}$. Find $\log h$.

A few variants...

The Multiple NFS

Considering multiple number fields.

$$
\begin{tikzcd}
 & & & \mathbb{Z}[X] \arrow[dlll] \arrow[dll] \arrow[d, "X \mapsto \theta_i"] \arrow[drr] \arrow[drrr] & & & \\
\mathbb{Q}(\theta_1) \arrow[drrr] & \mathbb{Q}(\theta_2) \arrow[drr] & \dots & \mathbb{Q}(\theta_i) \arrow[d, "\theta_i \mapsto m"] & \dots & \mathbb{Q}(\theta_{V-1}) \arrow[dll] & \mathbb{Q}(\theta_V) \arrow[dlll] \\
 & & & \mathbb{F}_{p^n} & & &
\end{tikzcd}
$$

- f_1, f_2 as in NFS
- $V - 2$ other polynomials; linear combinations of f_1, f_2.

The Tower NFS

$R = \mathbb{Z}[\iota]/h(\iota)$, h monic irreducible of degree n (more algebraic structure).

$$\begin{tikzcd}[column sep=small]
 & R[X] \arrow[dl] \arrow[dr] & \\
K_{f_1} \supset R[X]/(f_1(X)) \arrow[dr, "\alpha_{f_1} \mapsto m"] & & K_{f_2} \supset R[X]/(f_2(X)) \arrow[dl, "\alpha_{f_2} \mapsto m"] \\
 & R/\mathfrak{p} = \mathbb{F}_{p^n} &
\end{tikzcd}$$

The Special NFS

The characteristic p is the evaluation of a polynomial P of degree λ with small coefficients: $p = P(u)$ for $u \ll p$.

Example: BN family

- $P(z) = 36z^4 + 36z^3 + 24z^2 + 6z + 1$
- $u = -(2^{62} + 2^{55} + 1)$
- $p = P(u)$ (254 bits)

$$\boxed{p = \mathbf{16798108731015832284940804142231733909889187121439069848933715426072753864723}\ .}$$

The complexity of NFS and its variants

- 3 phases $=$ 3 costs $\longrightarrow$ overall complexity is sum of 3 costs.

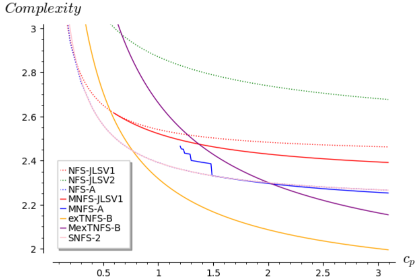
Goal: Optimize the maximum of these three costs.

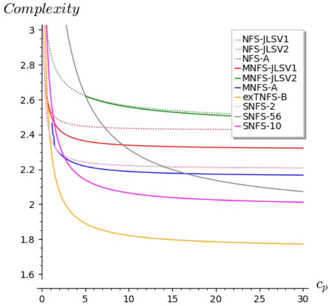
Why complicated?

1. Many parameters $\longrightarrow$ discrete or continuous, boundary issues.
2. Optimization problem $\longrightarrow$ Lagrange multipliers.
3. Solving a polynomial system $\longrightarrow$ Gröbner basis algorithm.
4. Uses many analytic number theory results.

A summary of these complexities

All complexities in $L_Q\left(1/3, c\right)$ for $p = L_Q\left(1/3, c_p\right)$.





Surprising facts:

- Not all the variants are applicable at the boundary case: STNFS has a much higher complexity!
- For small values of c_p, exTNFS better than MexTNFS.

What happens in small characteristics ?

The Function Field Sieve

$R = \mathbb{F}_p[\iota]$.

$$\begin{tikzcd}[column sep=small]
 & \mathbb{F}_p[X, Y] \arrow[dl, "Y \leftarrow g_2(X)"'] \arrow[dr, "X \leftarrow g_1(Y)"] & \\
\mathbb{F}_p[X] \arrow[dr, "X \leftarrow x"'] & & \mathbb{F}_p[Y] \arrow[dl, "Y \leftarrow y"] \\
 & \mathbb{F}_{p^n} &
\end{tikzcd}$$

- Using a <u>different mathematical object</u> (function fields).
- Similar to the special variant.

Quasi-polynomial algorithms

A lot of recent progress:

- 2013: complexity of $L_{p^n}(1/4 + o(1))$ (Joux)
- 2014: heuristic expected running time of $2^{O((\log \log p^n)^2)}$ (Barbulescu, Gaudry, Joux, Thomé)
- 2019: proven complexity! (Kleinjung and Wesolowski [KP19])

Theorem (Theorem 1.1 in [KP19])

Given any prime number p and any positive integer n, the discrete logarithm problem in the group $\mathbb{F}_{p^n}^{\times}$ can be solved in expected time $C_{QP} = (pn)^{2\log_2(n)+O(1)}$.

Lowering the complexity of FFS

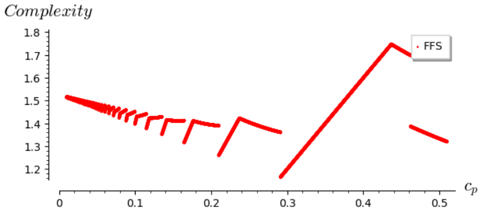
A shifted FFS

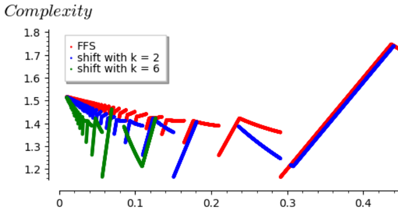
Our work: when $n = \kappa\eta$, we **lower** the complexity of FFS.

Main idea: work in a **shifted** finite field (similar to **Tower** setup)

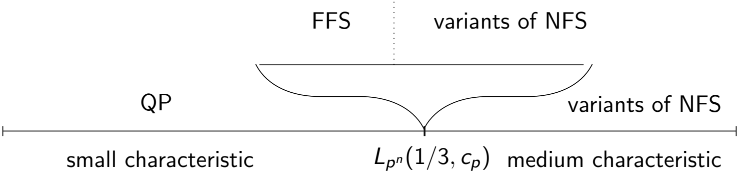
- Re-write: $\mathbb{F}_Q = \mathbb{F}_{p^n} = \mathbb{F}_{p^{\eta\kappa}} = \mathbb{F}_{p'^{\eta}}$, where $p' = p^\kappa$.
- From $p = L_Q(1/3, c_p)$, we get $p' = L_Q(1/3, \kappa c_p)$.

Complexity in $\mathbb{F}_{p^n}$ for $c_p = \alpha \Leftrightarrow$ complexity in $\mathbb{F}_{p'^{\eta}}$ at $c'_p = \kappa\alpha$.



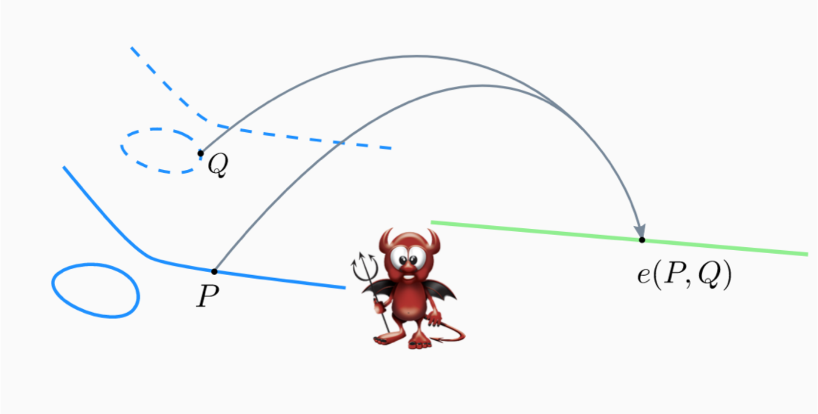


And the winners are ... !



For the variants of NFS, the best algorithm depends on considerations on n and p.

On the security of pairings



Constructing secure pairings

Asymptotically what finite field $\mathbb{F}_{p^n}$ should be considered in order to achieve the highest level of security when constructing a pairing?

Goal: find the optimal p and n that answers this question.

Did we study the correct area ?

Naive approach: $\sqrt{p} = L_Q\left(1/3, c_p\right)$.
More precise approach:

- Choose finite field where DLP is hard $\Rightarrow$ avoid QP area.

$$\boxed{p \geqslant \textbf{cross-over point between FFS and QP}}$$

- All the variants of FFS and NFS have a complexity in $L_Q(1/3, c)$: pick a finite field where the most efficient algorithm has the highest c.

  $\longrightarrow$ after our analysis, we can confirm that the highest complexities are indeed at $p = L_Q\left(1/3\right)$.

The ρ value in pairings

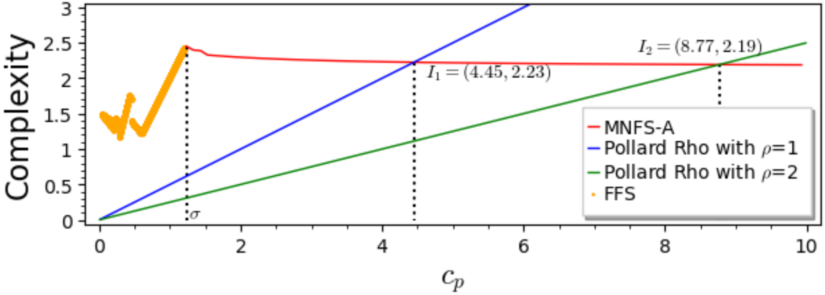
Consider a prime-order subgroup of $\mathcal{E}$ over $\mathbb{F}_p$ of size r.
Additional parameter: how large is this subgroup ?

$$\rho = \frac{\log p}{\log r}.$$

In all known construction: $\rho \in [1, 2]$.

(no efficient family of pairings asymptotically reaching $\rho = 1$.)

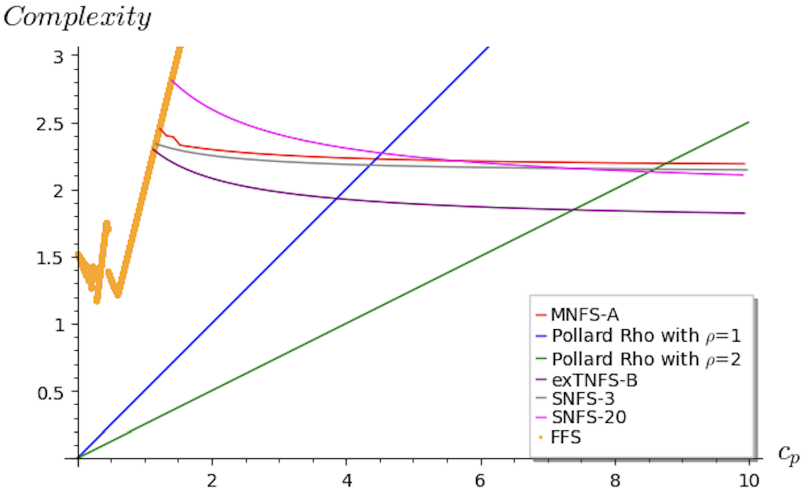
Goal: Look for value of c_p that maximizes $\min(\text{comp}_{\mathcal{E}}, \text{comp}_{\mathbb{F}_{p^n}})$.



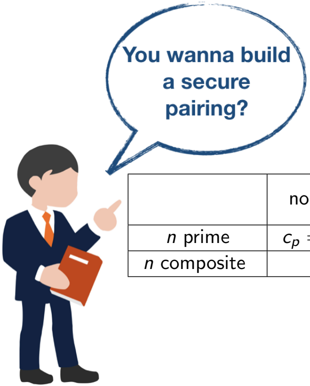
- Complexities for finite field DLP are decreasing functions.
- Pollard rho is an increasing function ($\text{complexity}_{\mathcal{E}} = p^{1/2\rho}$)

$\longrightarrow$ optimal c_p given by the intersection point!

When considering everyone!



Conclusion for pairings



	normal p	special p $\lambda = 20$	special p $\lambda = 3$
n prime	$c_p = 4.45,\ c_{\mathbf{MNFS}\text{-}\mathcal{A}} = 2.23$		$c_p = 4.36,\ c_{\mathbf{SNFS}\text{-}3} = 2.18$
n composite	$c_p = 3.93,\ c_{\mathbf{exTNFS}\text{-}\mathbf{B}} = 1.96$		

Suprising fact: Using a special form for p does not always make the pairing less secure ! It depends on the value of λ.

Thank you for your attention!

Questions?