# A Tale of Three Signatures: Practical Attack of ECDSA with wNAF

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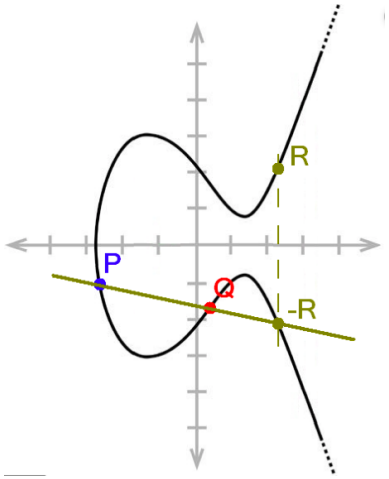
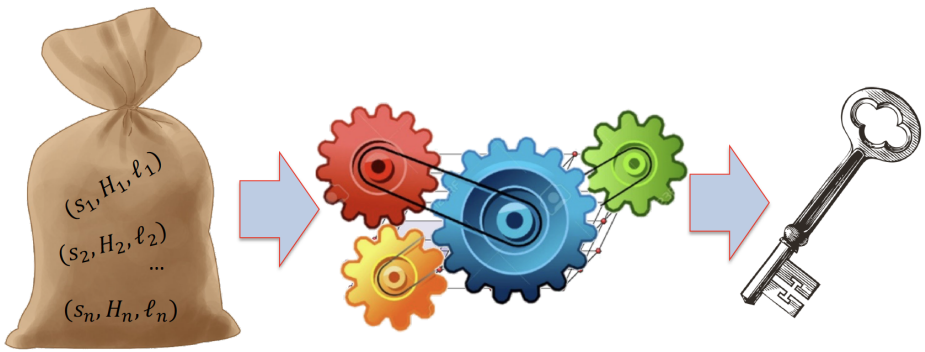
# How to attack ECDSA

1. Focus on the primitive: DLP on elliptic curves




2. OR get extra informations from an implementation: side channel attacks.



# Our work

- Improve the processing step of already known side-channel ECDSA attacks, using the Extended Hidden Number Problem and lattice techniques.
- Optimize the attack to maximize the success probability and minimize the overall time.
- $\longrightarrow$ Perform an attack with the minimum number of signatures needed to recover the secret key: only 3 signatures!

# Our target: ECDSA

**Elliptic Curve Digital Signature Algorithm** is a variant of the Digital Signature Algorithm, DSA, which uses elliptic curves instead of finite fields.

## Public Parameters

- An elliptic curve $E$ over a prime field.
- A generator $G$ of prime order $q$ on $E$.
- A hash function $H$ to $\mathbb{Z}_q$.

## Secret Key

- An integer $\alpha \in [1, q-1]$ .

## Public Key

- $p_k = [\alpha]G$: scalar multiplication of $G$ by $\alpha$.

# Signing algorithm

To sign a message $m$:

Step 1: Randomly select nonce $k \leftarrow_R \mathbb{Z}_q$

Step 2: Compute the point $(r, y) = [k]G$.

Step 3: Compute $s = k^{-1}(H(m) + \alpha r) \bmod q$.

Step 4: Output the signature $(r, s)$.

# Scalar multiplication

**Step 2: Compute the point $(r, y) = [k]G$**

$\longrightarrow$ **Scalar multiplication**

- Requires a fast algorithm
- Ideally that doesn't leak any information on $k$!

# Double-and-add algorithm

Goal: compute fast point multiplication on elliptic curves

- Input: integer $k$ and point $G$.
- Output: $Q = [k]G$

Step 1 : Convert $k$ to binary:
$k = k_0 + 2k_1 + 2^2 k_2 + \cdots + 2^t k_t$

Step 2 : Initialize $Q = \mathcal{O}$

Step 3 : For $j = t, \cdots, 0$, do:

- $Q \leftarrow 2Q$ double
- if $k_j = 1$: add $Q \leftarrow Q + G$

Step 4 : Return $Q$.

- Faster than repeated additions.
- Time of execution depends on number of 1s.
- Reduce Hamming weight of scalar $k$ $\longrightarrow$ (w)NAF representation.

# Non-adjacent form (NAF) and windowed-NAF (wNAF)

NAF:

- Impossible to have two consecutive non-zero digits,
- signed digits -1, 0, 1

wNAF:

- Impossible to have two consecutive non-zero digits,
- signed digits are in a larger window: $\in [-2^w + 1, 2^w - 1]$.

Example, 3 representations of 23:

- binary: $23 = 2^4 + 2^2 + 2^1 + 2^0 = (1, 0, 1, 1, 1)$
- NAF: $23 = 2^5 - 2^3 - 2^0 = (1, 0, -1, 0, 0, -1)$
- wNAF (for w=3): $23 = 2^4 + 7 \times 2^0 = (1, 0, 0, 0, 7)$

# wNAF in the wild

ECSDA with wNAF representation is used in:

- Bitcoin, as the signing algorithm for the transactions
- Some common libraries:
  - OpenSSL up to May 2019
  - Cryptlib
  - BouncyCastle
  - Apple's CommonCrypto





# Oh no! Information is being leaked!

**The power of side-channel attacks:**

Double and add is **not** constant time (depends on the number of non-zero coeff).

$\longrightarrow$ (Cache) timing attacks **identify** (most) of **the positions of the non-zero coefficients in the wNAF representation of the nonce** $k$**.**

Real $k$ (wNAF) representation (unknown from an attacker):

1 0 0 0 7 0 0 0 0 0 0 -7 0 0 0 0 0 0 3 0 0 0 0 0 0 0 5 0 0 0 0 0 0 0

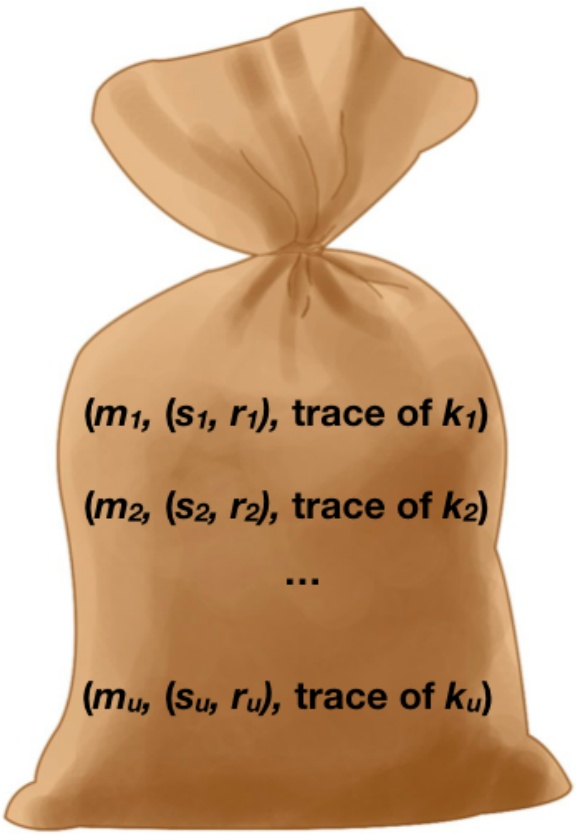
Information obtained by side channels:

$\star$ 0 0 0 $\star$ 0 0 0 0 0 0 $\star$ 0 0 0 0 0 0 $\star$ 0 0 0 0 0 0 0 $\star$ 0 0 0 0 0 0 0

# Information collected

What we have:

Many messages $m_i$ with their signatures $(s_i, r_i)$, signed by a unique secret key $\alpha$.




Side channels give the trace of $k_i$:

$\star\ 0\ 0\ 0\ \star\ 0\ 0\ 0\ 0\ \star\ 0\ 0\ 0\ \star\ 0\ 0\ 0\ \star\ 0\ 0\ 0$

The important information is:

- number of non-zero coefficients, $\ell_i$
- position of non-zero coefficients, $\lambda_1, \cdots, \lambda_{\ell_i}$

# The Extended Hidden Number Problem

Hlavác, Rosa (SAC 2007), *Extended hidden number problem and its cryptanalytic applications.*

Consider $u$ congruences of the form

$$a_i\alpha + \sum_{j=1}^{\ell_i} b_{i,j}k_{i,j} \equiv c_i \pmod{q},$$

- Unknowns: the secret $\alpha$ and $0 \leqslant k_{i,j} \leqslant 2^{\eta_{ij}}$,
- known values: modulus $q, \eta_{ij}$, $a_i$, $b_{i,j}$, $c_i$, $\ell_i$ for $1 \leqslant i \leqslant u$,

**Recover $\alpha$ in polynomial time.**

# Using EHNP to attack ECDSA

Goal: Transform ECDSA into an EHNP setup.

- ECDSA equation:

$$\alpha r = sk - H(m) \pmod{q}.$$

- Known information on the nonce $k$ :

$$k = \sum_{j=1}^{\ell} k_j 2^{\lambda_j} = \bar{k} + \sum_{j=1}^{\ell} d_j 2^{\lambda_j + 1},$$

- By substitution:

$$\alpha r_i - \sum_{j=1}^{\ell_i} 2^{\lambda_{i,j}+1} s_i d_{i,j} - (s_i \bar{k}_i - H(m_i)) \equiv 0 \pmod{q}$$

# The Extended Hidden Number Problem

We now have $u$ congruences of the form

$$a_i\alpha + \sum_{j=1}^{\ell_i} b_{i,j}k_{i,j} \equiv c_i \pmod{q},$$

given by

$$E_i : \quad \alpha r_i - \sum_{j=1}^{\ell_i} 2^{\lambda_{i,j}+1} s_i d_{i,j} - (s_i \bar{k}_i - H(m_i)) \equiv 0 \pmod{q}$$

- Unknowns: the secret key $\alpha$ and $0 \leqslant d_{i,j} \leqslant 2^{\mu_{i,j}}$,
- known values: modulus $q$, $r_i$, $\lambda_{i,j}$, $s_i$, $\bar{k}_i$, $\ell_i$, $H(m_i)$, $\mu_{i,j}$ for $1 \leqslant i \leqslant u$,

**Recover $\alpha$ in polynomial time.**

HOW? $\longrightarrow$ with lattices

# Reducing the size of the system

- We start with our system of modular equations $E_i$.
- Basic trick: Reduce the size of the system by eliminating $\alpha$ from the equations: $r_1 E_i - r_i E_1$
  - Remember that

$$\alpha = r_1^{-1} \left( \sum_{i=1}^{\ell_1} 2^{\lambda_{1,j}+1} s_1 d_{1,j} + (s_1 \bar{k}_1 - z_1) \right) \pmod{q}.$$

- New Goal: recover the $d_{i,j}$, with a new system of equations:

$$E_i' : \quad \sum_{j=1}^{\ell_1} \underbrace{\left(2^{\lambda_{1,j}+1} s_1 r_i\right)}_{:=\tau_{j,i}} d_{1,j} + \sum_{j=1}^{\ell_i} \underbrace{\left(-2^{\lambda_{i,j}+1} s_i r_1\right)}_{:=\sigma_{i,j}} d_{i,j} - \underbrace{r_1(s_i \bar{k}_i - H(m_i)) + r_i(s_1 \bar{k}_1 - H(m_1))}_{:=\gamma_i} \equiv 0 \pmod{q}.$$

# Lattice: Definition, bad and good bases

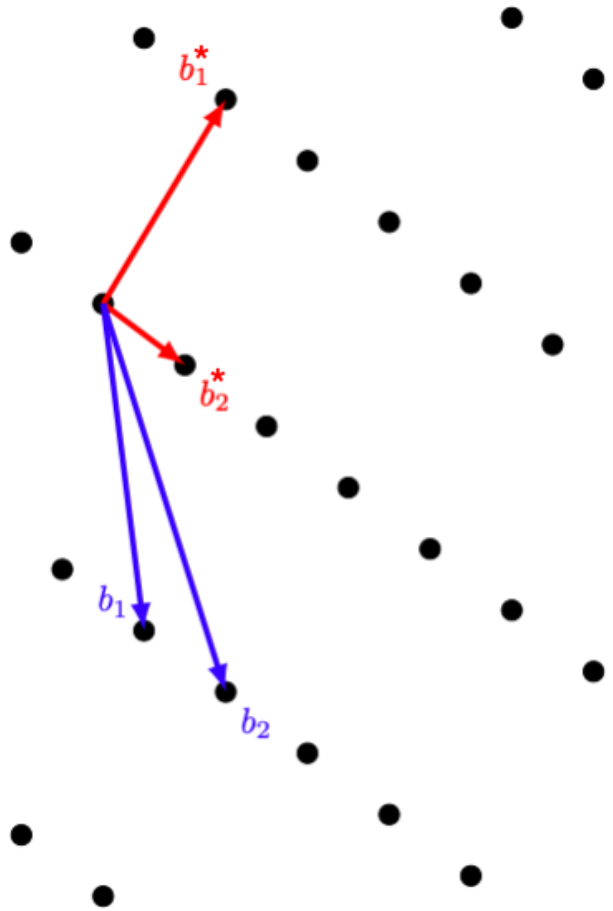
## Definition

A lattice is a discrete additive subgroup of $\mathbb{R}^n$, usually identified by a basis $\{b_1, \cdots, b_n\}$.

## Reduction algorithms: BKZ or LLL

Given an arbitrary basis $\{b_1, \cdots, b_n\}$, find a "better" basis $\{b_1^*, \cdots, b_n^*\}$.

Better $\rightarrow$ the first vectors are shorter (and more orthogonal) in the reduced basis.



# Our lattice construction

We construct a lattice such that there exists a linear combination $v$ of the lines containing the $d_{i,j}$:

$$v = (t_2, \cdots, t_u, d_{1,1}, \cdots, d_{u,\ell_u}, -1) \times \left(\begin{array}{cccc|ccccc} q & & & & & & & & \\ & \ddots & & & & & & & \\ & & \ddots & & & & & & \\ & & & q & & & & & \\ \hline E'_2 & E'_3 & \dots & E'_u & 2^{m-\mu_{1,1}} & & & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & & & \\ \vdots & \vdots & \vdots & \vdots & & & 2^{m-\mu_{u,\ell_u}} & & \\ & & & & 2^m & \dots & & 2^m & \end{array}\right)$$

$$v = (0, \dots, 0, d_{1,1}2^{m-\mu_{1,1}} - 2^{m-1}, \dots, d_{u,\ell_u}2^{m-\mu_{u,\ell_u}} - 2^{m-1}, -2^{m-1}).$$

# How to find v?

Goal: Find $v$.

- Good point: $v$ has a particular shape
-  It has no reason to appear in the basis
- $\longrightarrow$
  1. Make it short (by ugly manipulations of the lattice)
  2. Run BKZ on the basis[1]
  3. Pray to find a good shaped vector in the reduced basis
  4. Try to reconstruct $\alpha$ with the plausible $d_{i,j}$ you get.

[1] In practice $80 \leqslant \dim(\text{lattice}) \leqslant 215$.

# A new pre-processing method to speed-up the reduction

The slowest part of the attack: lattice reduction.
BKZ reduction time $\searrow$ if dimension $\searrow$ OR coefficients size $\searrow$.

Goal: Speed up the reduction time by $\searrow$ the size of the coefficients.

- Each trace $t$ comes with a notion of "weight" $\mu(t)$.
- Each coefficient of the basis is multiplied by $m = \max \mu(t)$ to get integer coefficients.
- The size of the coefficients depends on $m$.

Idea: $\longrightarrow$ **pre-select traces with small weight**

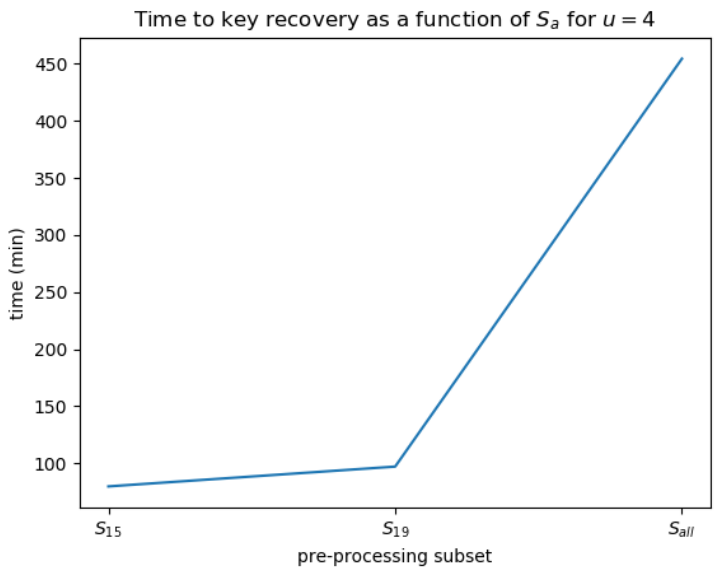
$$S_a = \{t \in \mathcal{T} | \mu(t) \leqslant a\}$$

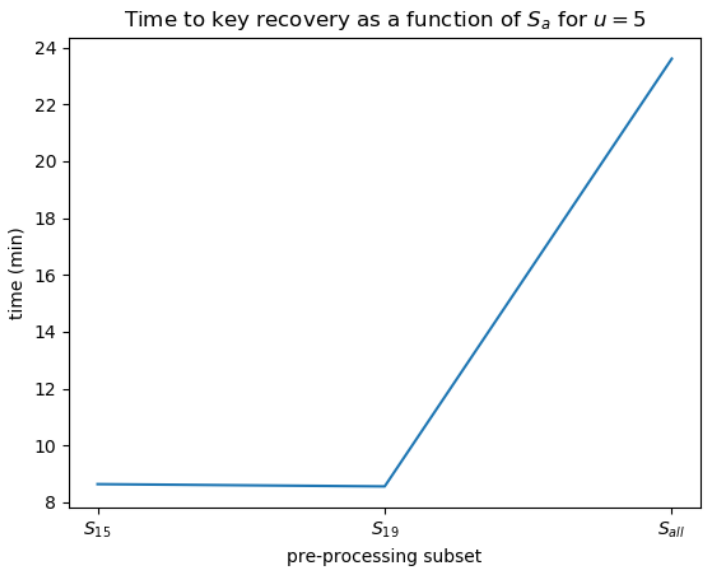
Numerical experiment: 5000 traces from OpenSSL: $a \in [11, 67]$.

# The effect of pre-processing

Key recovery time = time of 1 trial $\times$ nbr of trials to find the key.

- Considering 4 and 5 traces with BKZ-25.




- $S_{19}$: already 44% of the traces
- 3 traces: from 12 days ($S_{all}$) to 39 h ($S_{11}$) on a single core.

# 3 ways to evaluate the attack

Several parameters need to be balanced to mount an attack:

- the preprocessing subset of traces $S_a$, if any
- BKZ block size $\beta$: varies between 20 and 35
  - $\beta \nearrow \Rightarrow$ probability of success of 1 trial $\nearrow$ 
  - but $\beta \nearrow \Rightarrow$ reduction time $\nearrow$ 
- a multiplying coeff. in the lattice

What is the minimal amount of signatures an attacker can use?

What are the parameters that lead to

- the fastest attack?
- the best probability of success?

# Our Main Results

- 3 signatures: 39 hours, small probability of success, $S_{11}$, BKZ-35.
- Our fastest attack:
  - 4 signatures: 1 hour 17 minutes, BKZ-25, $S_{15}$
  - 8 signatures: 2 minutes 25 seconds, BKZ-20, $S_{all}$
- Our most successful attack:
  - 4 signatures: 4% of success per trial, BKZ-35, $S_{all}$
  - 8 signatures: 45% of success per trial, BKZ-35, $S_{all}$

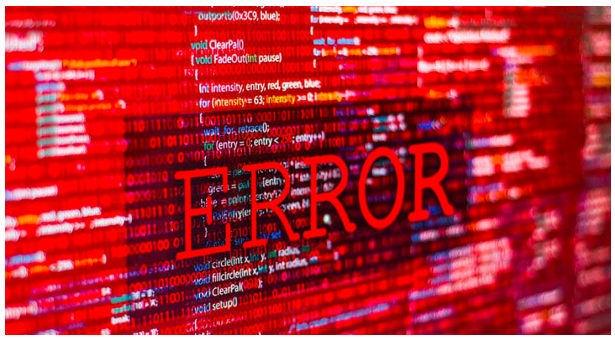
# Previous attacks on ECDSA with wNAF

- **Comparing with another variant of EHNP**
  Fan, Wang, Cheng (CCS 2016), *Attacking OpenSSL implementation of ECDSA with a few signatures*

| Attack | # signatures | Probability of success | Overall time |
|---|---|---|---|
| [FWC2016] | 5 | 4% | 15 hours/18 minutes |
| | 6 | 35% | 1 hour 21 minutes/18 minutes |
| | 7 | 68% | 2 hours 23 minutes/34.5 minutes |
| Our attack | 3 | 0.2% | 39 hours |
| | 4 | 4% | 1 hour 17 minutes |
| | 5 | 20% | 8 minutes 20 seconds |
| | 6 | 40% | 5 minutes |
| | 7 | 45% | 3 minutes |
| | 8 | 45% | 2 minutes |

- **Comparing with the Hidden Number Problem**
  Van de Pol, Smart, Yarom (CT-RSA 2015) *Just a Little Bit More.* 13 signatures, 54% probability of success and 21 seconds total time to key recovery.

# Errors can occur, and they often do!




**Side-channel analyzis is not perfect.**

Real $k$ (wNAF) representation (unknown from an attacker):

1 0 0 0 7 0 0 0 0 0 0 -7 0 0 0 0 0 0 3 0 0 0 0 0 0 0 5 0 0 0 0 0 0 0

Information obtained by side channels:

$\star$ 0 0 0 $\star$ 0 0 0 0 0 0 $\star$ 0 0 0 0 0 0 0 0 0 0 0 0 0 $\star$ 0 0 0 0 0 0 0

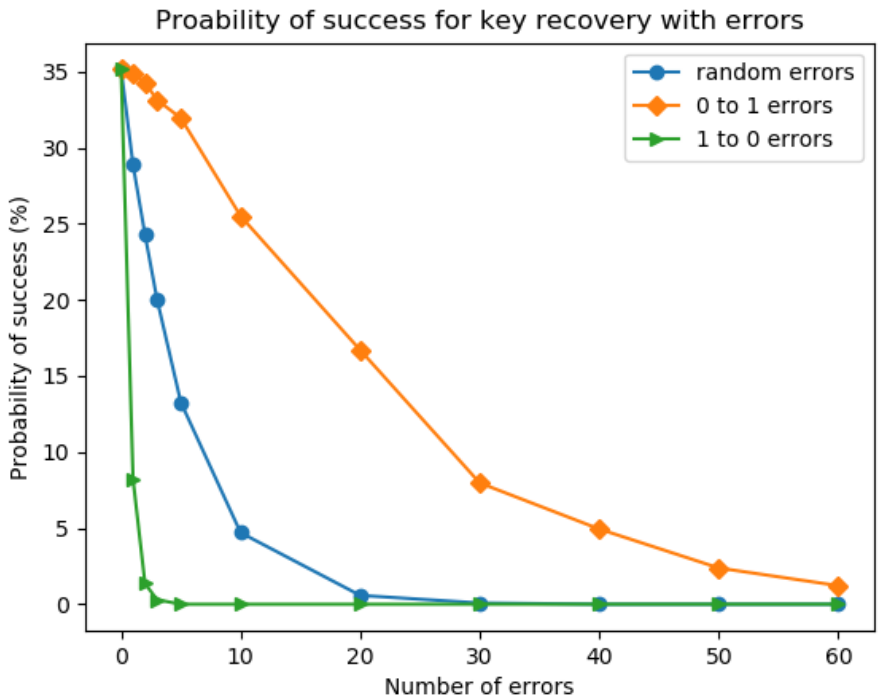
# Probability of success with various types of error

## Error type 1:

A 0 coefficient misread as $\star$: adds a new variable to the system, the nbr of non-zero digits is overestimated.

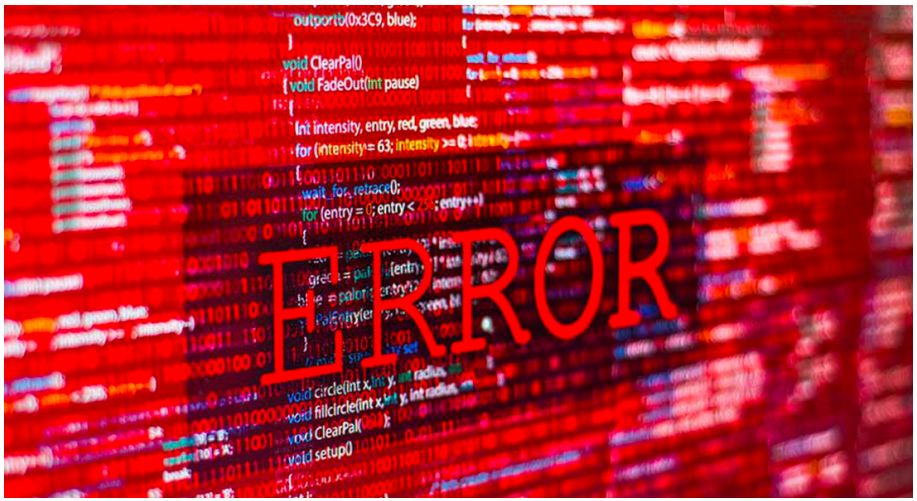
## Error type 2:

A non-zero coefficient misread as 0: lose information necessary for key recovery.




**Error 2 affects the probability of success of key recovery much more.**

# Resilience up to 2% of errors




- Morality: Resilience to errors up to 2% of misread digits.
- Resilience increase to 4% if we avoid certain types of errors.
- Strategy: in the side channel part, if you are not confident about your reading, choose to put a $\star$ instead of a 0.

# Thank you!

A Tale of Three Signatures: practical attack of ECDSA with wNAF
Gabrielle De Micheli, Cécile Pierrot, Rémi Piau
https://eprint.iacr.org/2019/861

# Fastest attack

| Number of signatures | Total time | Parameters | | | Probability of success (%) |
|---|---|---|---|---|---|
| | | BKZ | Preprocessing | $\Delta$ | |
| 3 | 39 hours | 35 | $S_{11}$ | $\approx 2^3$ | 0.2 |
| 4 | 1 hour 17 | 25 | $S_{15}$ | $\approx 2^3$ | 0.5 |
| 5 | 8 min 20 | 25 | $S_{19}$ | $\approx 2^3$ | 6.5 |
| 6 | 3 min 55 | 20 | $S_{all}$ | $\approx 2^3$ | 7 |
| 7 | 2 min 43 | 20 | $S_{all}$ | $\approx 2^3$ | 17.5 |
| 8 | 2 min 25 | 20 | $S_{all}$ | $\approx 2^3$ | 29 |

Total time key recovery = time of single trial $\times$ number of trials to find the key.

# Highest probability of success of a single trial

| Number of signatures | Probability of success (%) | Parameters | | | Total time |
|---|---|---|---|---|---|
| | | BKZ | Preprocessing | $\Delta$ | |
| 3 | 0.2 | 35 | $S_{11}$ | $\approx 2^3$ | 39 hours |
| 4 | 4 | 35 | $S_{all}$ | $\approx 2^3$ | 25 hours 28 |
| 5 | 20 | 35 | $S_{all}$ | $\approx 2^3$ | 2 hours 42 |
| 6 | 40 | 35 | $S_{all}$ | $\approx 2^3$ | 1 hour 04 |
| 7 | 45 | 35 | $S_{all}$ | $\approx 2^3$ | 2 hours 36 |
| 8 | 45 | 35 | $S_{all}$ | $\approx 2^3$ | 5 hours 02 |

# Comparing times with Fan et al, CCS 2016

| Number of signatures | Our attack | | Fan et al | |
|---|---|---|---|---|
| | Time | Success (%) | Time | Success (%) |
| 3 | 39 hours | 0.2% | – | – |
| 4 | 1 hour 17 minutes | 0.5% | 41 minutes | 1.5% |
| 5 | 8 minutes 20 seconds | 6.5% | 18 minutes | 1% |
| 6 | $\approx$ 5 minutes | 25% | 18 minutes | 22% |
| 7 | $\approx$ 3 minutes | 17.5% | 34 minutes | 24% |
| 8 | $\approx$ 2 minutes | 29% | – | – |

# Comparing success probabilities with Fan et al, CCS 2016

| Number of signatures | Our attack | | Fan et al | |
|---|---|---|---|---|
| | Success (%) | Time | Success (%) | Time |
| 3 | 0.2% | 39 hours | – | – |
| 4 | 4% | 25 hours 28 minutes | 1.5% | 41 minutes |
| 5 | 20% | 2 hours 42 minutes | 4% | 36 minutes |
| 6 | 40% | 1 hour 4 minutes | 35% | 1 hour 43 minutes |
| 7 | 45% | 2 hours 36 minutes | 68% | 3 hours 58 minutes |
| 8 | 45% | 5 hours 2 minutes | – | – |

# Error analysis using BKZ-25, $\Delta \approx 2^3$ and $S_{all}$.

| Number of signatures | Probability of success (%) 0 errors | 5 errors | 10 errors | 20 errors | 30 errors |
|---|---|---|---|---|---|
| 4 | 0.28 | $\ll 1$ | 0 | 0 | 0 |
| 5 | 4.58 | 0.86 | 0.18 | $\ll 1$ | 0 |
| 6 | 19.52 | 5.26 | 1.26 | 0.14 | $\ll 1$ |
| 7 | 33.54 | 10.82 | 3.42 | 0.32 | $\ll 1$ |
| 8 | 35.14 | 13.26 | 4.70 | 0.58 | $\ll 1$ |

- Corresponds to a resilience of 2% of errors.
- Total time: 1 out of 5000 experiments, 46 sec per experiment, 65 hours on a single core